Double inequality related to Hyperfactorial.
https://www.linkedin.com/feed/update/urn:li:activity:7240653679865806848?utm_source=share&utm_medium=member_desktop
If e is Euler's number, prove that:

$$(n+1)^{-n-1}e^{\frac{n(n+1)}{2}} < 1^{1-n}2^{3-n} \cdot 3^{5-n} \cdot \ldots \cdot n^{n-1} \le e^{\frac{(n-1)n}{2}}.$$

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Let $P_n := \prod_{k=1}^{n} k^{2k-1-n}, n \in \mathbb{N}$ and $H(n) := \prod_{k=1}^{n} k^k$ (hyperfactorial). Since $P_n = \prod_{k=1}^{n} \frac{k^{2k}}{k^{n+1}} = \frac{1}{(n!)^{n+1}} \prod_{k=1}^{n} k^{2k} = \frac{H^2(n)}{(n!)^{n+1}}$ then $(n+1)^{-n-1}e^{\frac{n(n+1)}{2}} < P_n \le e^{\frac{(n-1)n}{2}} \Leftrightarrow$

$(n+1)^{-n-1}e^{\frac{n(n+1)}{2}} < \frac{H^2(n)}{(n!)^{n+1}} \le e^{\frac{(n-1)n}{2}} \Leftrightarrow (n+1)^{-n-1}e^{\frac{n(n+1)}{2}} < \frac{H^2(n)}{(n!)^{n+1}} \le e^{\frac{(n-1)n}{2}} \Leftrightarrow$

$$\left(\frac{n!}{n+1}\right)^{n+1} \cdot e^{\frac{n(n+1)}{2}} < H^2(n) \le e^{\frac{(n-1)n}{2}} \cdot (n!)^{n+1}.$$

Since $\left(\frac{n!}{n+1}\right)^{n+1} e^{\frac{n(n+1)}{2}} < H^2(n) \Leftrightarrow$

$$\left(\frac{n!}{n+1}\right)^{n+1} e^{\frac{n(n+1)}{2}} \cdot (n+1)^{2(n+1)} < H^2(n) \cdot (n+1)^{2(n+1)} \Leftrightarrow$$

$$((n+1)!)^{n+1} \cdot e^{\frac{n(n+1)}{2}} < H^2(n+1), n \in \mathbb{N}$$

and $(1!)^1 \cdot e^{\frac{(1-1)\cdot 1}{2}} = H^2(1)$ then our problem reduced to prove inequality

(1) $\quad L(n) \le H^2(n) \le U(n), \forall n \in \mathbb{N},$ where

$L(n) := (n!)^n \cdot e^{\frac{(n-1)n}{2}}$ and $U(n) := (n!)^{n+1} \cdot e^{\frac{(n-1)n}{2}}$.

First we will prove that

(2) $\quad \frac{L(n+1)}{L(n)} < \frac{H^2(n+1)}{H^2(n)} < \frac{U(n+1)}{U(n)}$ any $n \in \mathbb{N}$.

We have (**2**) $\Leftrightarrow \frac{((n+1)!)^{n+1} \cdot e^{\frac{n(n+1)}{2}}}{(n!)^n \cdot e^{\frac{(n-1)n}{2}}} < (n+1)^{2(n+1)} < \frac{((n+1)!)^{n+2} e^{\frac{n(n+1)}{2}}}{(n!)^{n+1} e^{\frac{(n-1)n}{2}}} \Leftrightarrow$

$\begin{cases} (n+1)^n < n!e^n \\ n! \cdot e^n < (n+1)^{n+1} \end{cases} \Leftrightarrow \frac{(n+1)^n}{e^n} < n! < \frac{(n+1)^{n+1}}{e^n}$, where the latter double inequality holds* for any $n \in \mathbb{N}$.

Since $L(1) = H^2(1) = U(1)$ and $\frac{L(n+1)}{L(n)} < \frac{H^2(n+1)}{H^2(n)} < \frac{U(n+1)}{U(n)}$ for $\forall n \in \mathbb{N}$ then by MI $L(n) \le H^2(n) \le U(n), \forall n \in \mathbb{N}$ (with equality in both sides iff $n = 1$).

Indeed, for any $n \in \mathbb{N}$ assuming $L(n) \le H^2(n) \le U(n)$ and using inequality (**2**) we obtain $L(n+1) = \frac{L(n+1)}{L(n)} \cdot L(n) < \frac{H^2(n+1)}{H^2(n)} \cdot H^2(n) = H^2(n+1)$

and $H^2(n+1) = \frac{H^2(n+1)}{H^2(n)} \cdot H^2(n) < \frac{U(n+1)}{U(n)} \cdot U(n) = U(n+1)$.

* Since $e < \left(1 + \frac{1}{n}\right)^{n+1} = \frac{(n+1)^{n+1}}{n^{n+1}}, \forall n \in \mathbb{N}$ then $e^n < \prod_{k=1}^{n} \frac{(k+1)^{k+1}}{k^{k+1}} =$
$\frac{1}{n!} \prod_{k=1}^{n} \frac{(k+1)^{k+1}}{k^k} = \frac{(n+1)^{n+1}}{n!} \Leftrightarrow n! < \frac{(n+1)^{n+1}}{e^n}$.
Inequality $\frac{(n+1)^n}{e^n} < n! \Leftrightarrow (n+1)^n < e^n \cdot n!$ we will prove using MI.
For $n = 1$ we have $(1+1)^1 < 1!e^1 \Leftrightarrow 2 < e$ and
$\frac{(n+2)^{n+1}}{(n+1)^n} < \frac{(n+1)!e^{n+1}}{n!e^n} = (n+1)e \Leftrightarrow \frac{(n+2)^{n+1}}{(n+1)^{n+1}} < e \Leftrightarrow$
$\left(1 + \frac{1}{n+1}\right)^{n+1} < e, \forall n \in \mathbb{N}$.
For any $n \in \mathbb{N}$ assuming $(n+1)^n < e^n n!$ we obtain
$(n+2)^{n+1} = \frac{(n+2)^{n+1}}{(n+1)^n} \cdot (n+1)^n < \frac{(n+1)!e^{n+1}}{n!e^n} \cdot e^n n! = (n+1)!e^{n+1}$.
Thus, by MI $(n+1)^n \leq n!e^n, \forall n \in \mathbb{N}$.
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